Technology and Production

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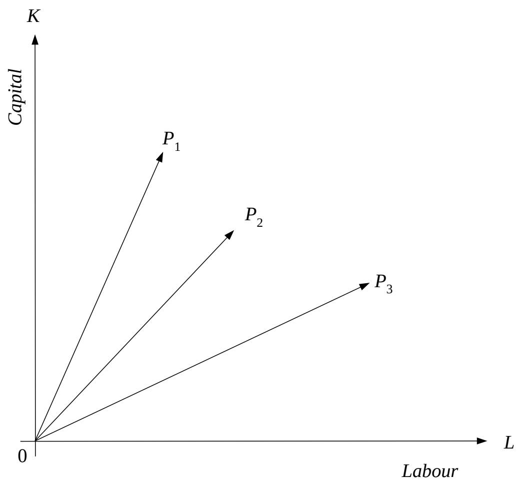
Isoquants and Technical Efficiency

In neoclassical economics, the production function is a purely technical relationship between inputs and outputs. Hence a production function identifies the technology available to a producer. Similarly, it has been argued that it is legitimate to aggregate across firms and/or industries to produce aggregate production functions.

At its simplest, we can conceive of a series of production process whereby different quantities of inputs can be used to produce a unit of a given commodity, e.g., in a two-input world.

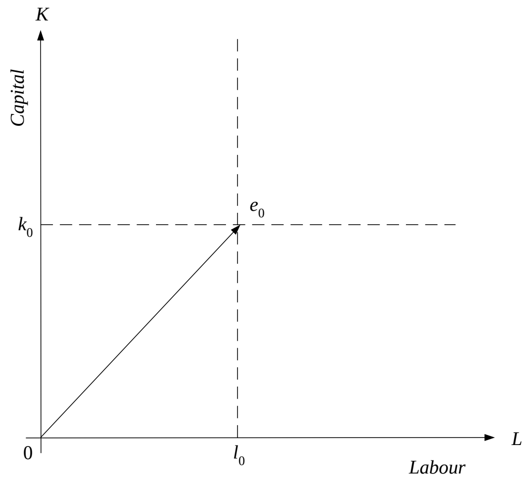
Figure 2.1 Production Processes



In this illustration, the processes observed use progressively less capital and progressively more labour. Hence, none uses the <u>same</u> amount of one input and more of another. This is deliberate, since typically a production function only identifies <u>technically</u> efficient input combinations, i.e., combinations that use less of one factor and no more of the other factors. Note this means that processes that use more of one factor and less of another factor cannot be directly compared.

This can be seen in a simple diagram of production processes.

Figure 2.2 Production Processes and Technical Efficiency



Using k_0 and l_0 of inputs to produce a fixed quantity of output is defined as technically efficient. Hence, any combination above and to the right of e_0 would be technically inefficient and a combination below and to the left of e_0 would technically be feasible. But any combination above and to the left of e_0 and below to the right of e_0 cannot be compared because they use less of one factor and more of another. Notice the similarity to the indifference curve derivation and revealed preferences.

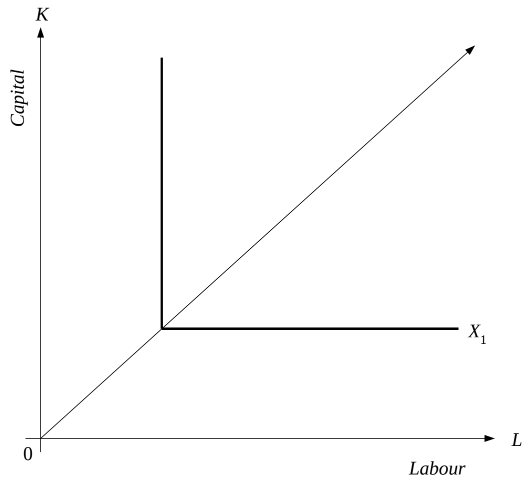
In fact, we can derive a set of relations for producers that are similar to those for consumers. These are called isoquants as opposed to indifference curves.

Production Processes and Isoquants

<u>The Leontief Isoquant</u>

Assume there is a single production process, i.e.,

Figure 2.3 The Leontief Isoquant

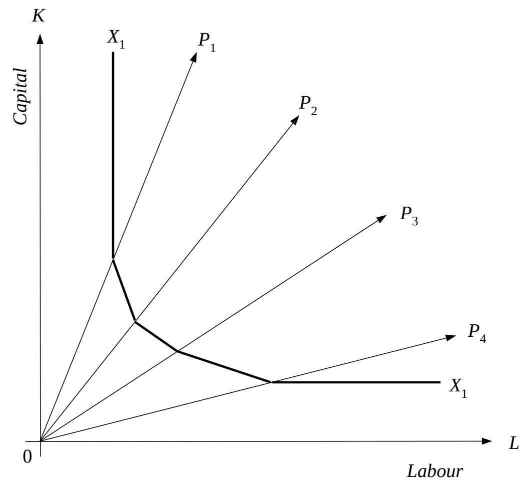


then for any level of output we have, from our definition of technical efficiency, a right-angled isoquant, i.e., there is strict complementarity (no substitution possibilities) between the factors. This type of isoquant is implicit in input-output analysis.

Linear-Programming Isoquant

Assume now there are a limited number of production processes, and that these can be combined.

Figure 2.4 Linear-Programming Isoquant

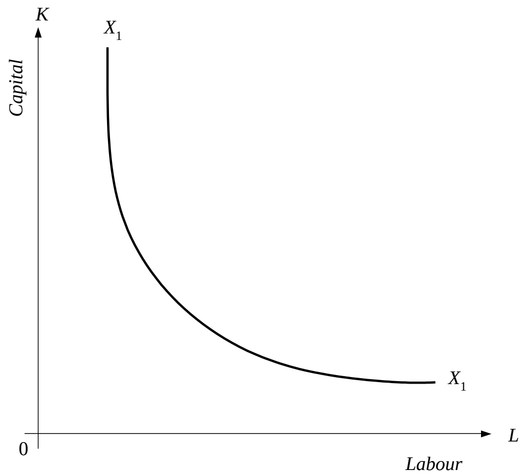


Then for any level of output we have a kinked isoquant where the slope becomes flatter as we move from left to right. Note that the segments of the isoquants are linear, which implies the assumptions that processes can be combined and are divisible.

Smooth Convex Isoquant

Assume now that there are an infinite number of processes. Then, by the definition of technical efficiency, we get a smooth convex isoquant, i.e.,

Figure 2.4 Smooth Convex Isoquant



The kinked, or linear-programming isoquant, is probably more realistic, but a large proportion of analyses assumes a smooth convex isoquant for two reasons:

i) it permits the easy use of calculus;

ii) it approximates a kinked isoquant.

The production function does not define a single isoquant; rather it defines a whole array of isoquants with one for each level of output. Thus, isoquants are everywhere dense and, because of our definition of technical efficiency, cannot intersect.

The Production Function

The general form of a production function is

$$Y = f\left(L, K, R, N, \upsilon, \gamma\right) \tag{1}$$

where Y = output;

L = labour;

K = capital;

R = raw materials;

N = land;

υ = returns to scale;

γ = an efficiency parameter.

but we will concern ourselves with a slightly simpler form. Let us define output as value added, X, where

$$X = Y - R. \tag{2}$$

This undermines the purely technical nature of the relationship since Y and R are measured in non-commensurate units, and thus value added is measured in money units. We will also ignore land, or alternatively, treat it as part of capital. Thus, we get the following general form

$$X = f(L, K, \upsilon, \gamma) \tag{3}$$

or, in a specific form which we will use

$$X = aL^{\alpha}K^{\beta} \tag{4}$$

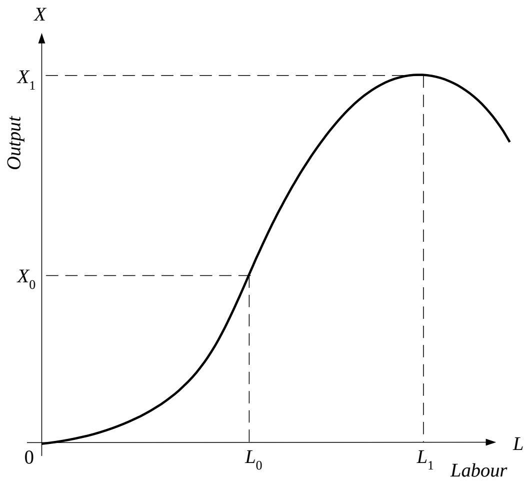
that is a Cobb-Douglas production function where a = the efficiency parameter; α, β = coefficients.

Although the scale factor has apparently disappeared it will soon be seen that it is related to the coefficients. You should remind yourself how to differentiate both the general and Cobb-Douglas forms of production function.

<u>Short-run Production Functions and Marginal Products</u>

Throughout we will assume that labour is variable in the short run and capital is only variable in the long run. Short-run production functions refer to the situation where one factor is variable and all others are fixed, e.g.,

Figure 2.5 Short-Run Production Function



Thus, as more labour is used with a fixed quantity of capital, and fixed returns to scale and efficiency, output increases to a peak, with L_1 of labour, and then declines. Note also that output is assumed initially to climb progressively more rapidly but the rate of output growth then gradually declines.

The change in output resulting from a change in labour input, *ceteris paribus*, is the marginal product of labour, i.e.,

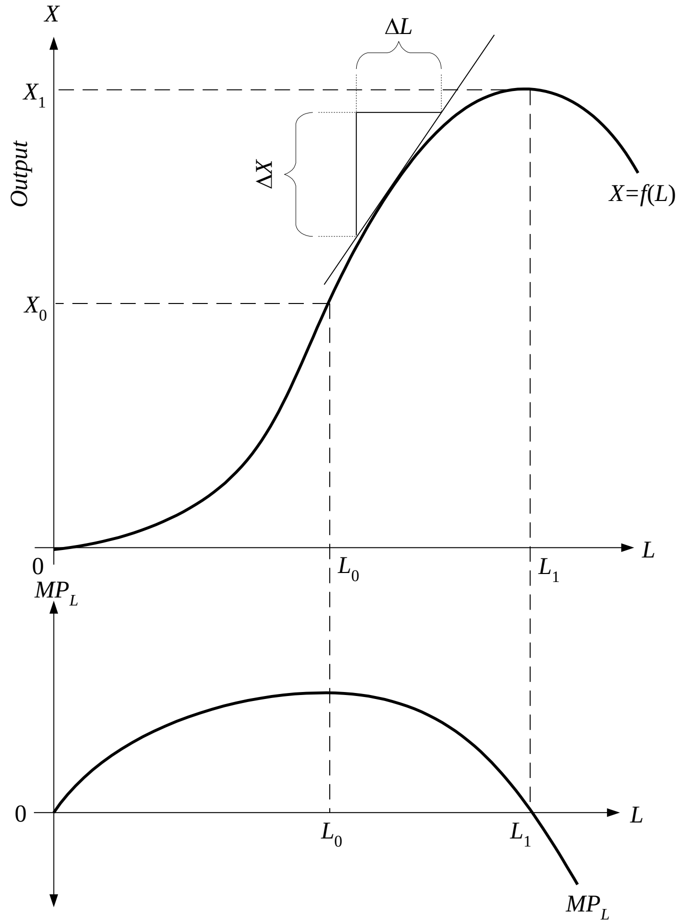
$$MP_L = \frac{\partial X}{\partial L} \tag{5}$$

and is the slope of the production function

$$X = f\left(L\right)_{\bar{K},\bar{\upsilon},\bar{\gamma}} \tag{6}$$

Thus, we can derive MP_L curve from the short-run production function.

Figure 2.6 Short-Run Production Function & Marginal Products



Note that the MP_L curve has three distinct phases:

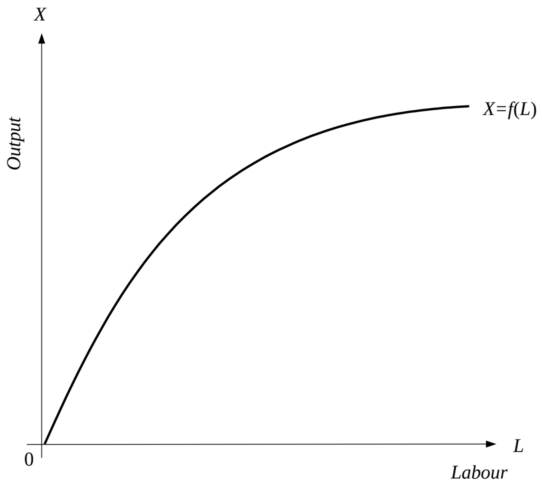
i) $$\frac{\partial X}{\partial L} = MP_L > 0 \text{ and } \frac{\partial^2 X}{\partial L^2} = \frac{\partial (MP_L)}{\partial L} > 0$$

ii) $$\frac{\partial X}{\partial L} = MP_L > 0 \text{ and } \frac{\partial^2 X}{\partial L^2} = \frac{\partial (MP_L)}{\partial L} < 0$$

iii) $$\frac{\partial X}{\partial L} = MP_L < 0 \text{ and } \frac{\partial^2 X}{\partial L^2} = \frac{\partial (MP_L)}{\partial L} < 0$$

If, it is sensible to produce any output, then during phase 1 it would be irrational not to employ **more** labour since this increases the MP_L by more than the previous unit of labour. Similarly, in phase 3 it would be irrational not to employ **less** labour, since this would increase output. Hence, it is only rational to employ labour over the range defined by phase 2. Thus, we typically find the short-run production is drawn as

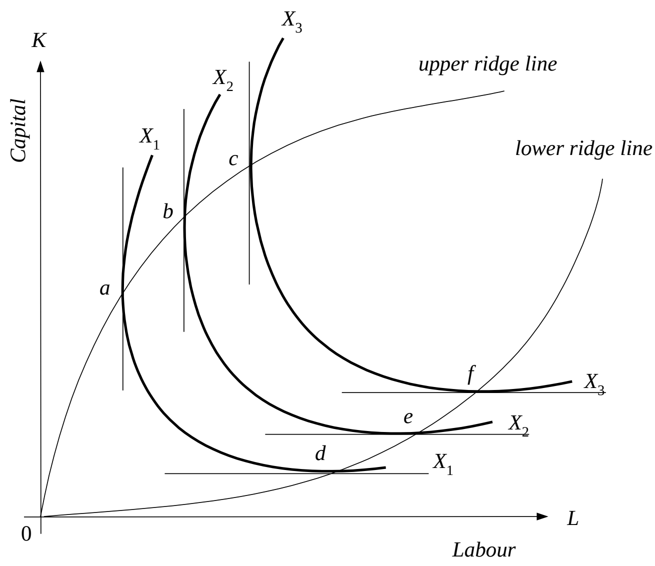
Figure 2.7 Standard Short-Run Production Function



where the slope, i.e., MP_L, is strictly positive.

This result is important, not only because of its definition of rationality, but because it defines the 'range' of the isoquants, i.e.,

Figure 2.8 Upper & Lower Ridge Lines



At *a*, *b* and *c* the $MP_K = 0$, since 'beyond' *a*, *b* and *c* the $MP_K < 0$, while 'before' *a*, *b* and *c* the $MP_K > 0$. Similarly, at *d*, *e* and f the $MP_L = 0$, whereas 'beyond' *d*, *e* and f the $MP_L < 0$ and 'before' *d*, *e* and f the $MP_L > 0$.

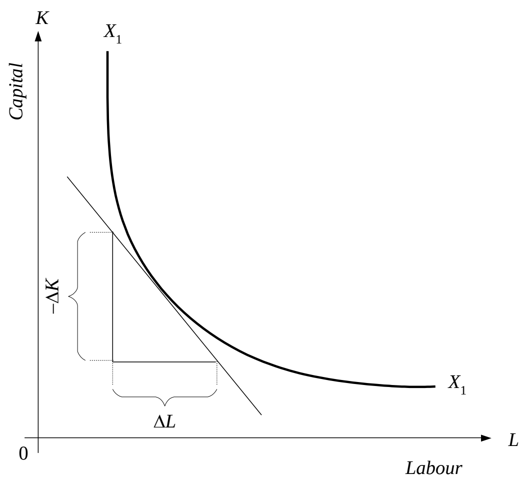
Thus, only the range over which isoquants are convex to the origin satisfies the condition of positive but declining marginal products.

It is important to note that the vast majority of functional forms used in economic models have properties that ensure the isoquants are always convex to the origin, i.e., are not 'backward bending', and hence that the MPs are strictly positive and decreasing. Where this is not the case applications usually impose additional conditions that ensure the isoquants are convex to the origin.

<u>Marginal Rate of Technical Substitution (MRTS)</u>

To see this more clearly we need to develop the concept of *MRTS*. As with indifference curves, the slope of the isoquant defines the degree of substitutability, i.e.,

Figure 2.9 Marginal Rate of Technical Substitution



thus, we get

$$-\frac{\partial K}{\partial L} = MRTS_{L,K} \quad (7)$$

the marginal rate of technical substitution of labour <u>for</u> capital. As with the MRS, the MRTS is equal to the ratio of the MP of the factors.

Let the production function be

$$X = f(L,K) \quad (8)$$

then

$$dX = \left(\frac{\partial X}{\partial K}\right).\partial K + \left(\frac{\partial X}{\partial L}\right).\partial L = 0 \tag{9}$$

along an isoquant, Thus

$$-\frac{\partial K}{\partial L} = \frac{(\partial X/\partial L)}{(\partial X/\partial K)} = \frac{MP_L}{MP_K} = MRTS_{L,K} \tag{10}$$

and on the lower labour ridge line

$$MRTS_{L,K} = \frac{(\partial X/\partial L)}{(\partial X/\partial K)} = \frac{0}{(\partial X/\partial K)} = 0 \tag{11}$$

Elasticity of Substitution (Technical)

A major weakness of the *MRTS* is its dependence upon the units used to measure inputs. To avoid this an elasticity measure is preferable.

The elasticity of substitution, σ (sigma) is defined as

$$\begin{aligned}\sigma &= \frac{\%\Delta \text{ in } K/L}{\%\Delta \text{ in } MRTS} \\ &= \frac{dK/L \big/ K/L}{dMRTS \big/ MRTS}\end{aligned} \tag{12}$$

which is a pure number since both numerators and denominators are measured in the same units.

Optimal Choice of Factor Combinations

Derivation of production functions is OK, but of little use alone. All the production function is doing is identifying technically efficient input combinations. But, while a producer/firm will be interested in operating a technically efficient plant, she will also be interested in maximising profit, i.e., will also wish to be allocatively efficient. By being both technically and allocatively efficient, she will achieve economic efficiency.

So, what is allocative efficiency? It is about choosing the optimum combination of technically efficient inputs, and is in fact a constrained optimisation problem which can be phrased in one of two ways:

i) maximisation of output subject to a cost constraint; or

ii) minimisation of cost subject to a given (fixed output.

In fact, both can be written as variants of a profit maximisation problem where the firm's objective is profit maximisation subject to different constraints, i.e.,

Max output

$$\begin{aligned} \max \quad \Pi &= R - C \\ &= \overline{P}_x X - \overline{C} \end{aligned} \tag{17}$$

where $\overline{P}_x$ = output prices
$\overline{C}$ = total (given) costs

Min Cost

$$\begin{aligned} \max \quad \Pi &= R - C \\ &= \overline{P}_x \overline{X} - C \end{aligned} \tag{18}$$

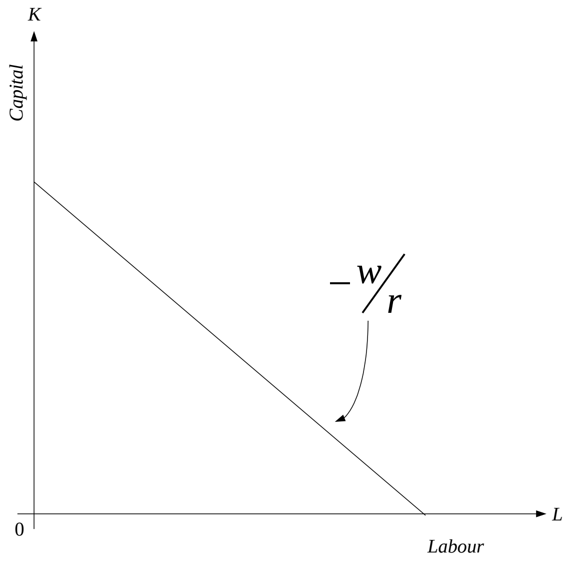
We will derive the case for maximising output subject to a cost constraint, but you should work through the other case yourselves. First, we need however to develop the concept of an isocost line.

Let assume there are two inputs, K and L, and their supply curves to the firm are perfectly elastic, i.e., the firm can buy any quantities of K and L at constant prices, where r = price per unit of capital services and w = wage rate. Then

$$C = wL + rK \tag{19}$$

and an isoccost line can be defined as the locus of all combinations of input with the same total cost. Thus

Figure 2.10 Isocost Line



and the slope of the isocost line is

$$= -\frac{w}{r} \qquad (20)$$

since we can rearrange the cost constraint to give

$$K = \frac{\bar{C}}{r} - \frac{w}{r}.L \qquad (21)$$

and thus, the slope is equal to the ratio of factor prices.

<u>Maximisation of Output Subject to a Cost Constraint</u>

For a given production function

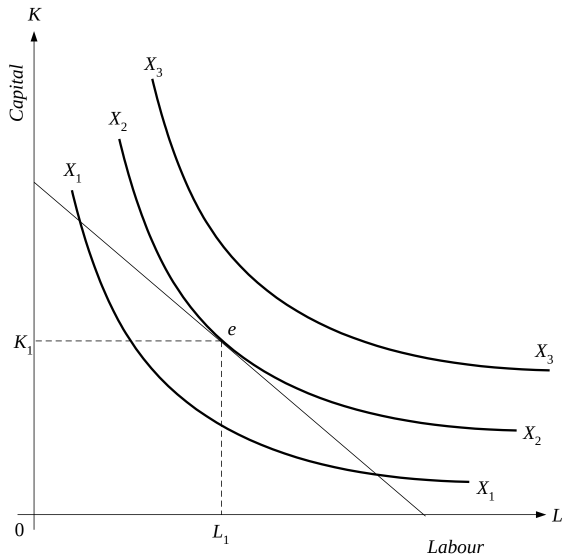
$$X = f\left(L, K, \upsilon, \gamma\right) \qquad (22)$$

and a cost constraint

$$\bar{C} = \bar{w}L + \bar{r}K \qquad (23)$$

we can derive the following diagram

Figure 2.11 Maximisation of Output Subject to a Cost Constraint



and the slope of the isocost line is $-w/r$.

All isoquants are technically efficient combinations. But X_3 cannot be achieved because of the cost constraint, $\overline{C}$. Further, output level X_1 would not be an output maximisation decision given $\overline{C}$, since X_2 could be produced. Finally, X_2 can be produced, but only by using L_1 and K_1 of inputs.

Further, at *e*, slope of isocost line = slope of the isoquant, i.e.,

$$-\frac{w}{r} = -\frac{\partial K}{\partial L} = MRTS_{L,K} = \frac{MP_L}{MP_K} . \tag{24}$$

That is, the ratio of input prices is equal to the ratio of *MRTS*. This is the condition for allocative efficiency. And the simultaneous achievement of allocative and technical efficiency yields economic efficiency.

Formally this is a constrained optimisation problem that is virtually identical to those found in consumer theory. That is

$$\begin{aligned} &\text{Maximise} && X = f(K,L) \\ &\text{sto} && \overline{C} = \overline{w}L + \overline{r}K \\ &\text{or} && \overline{C} - \overline{w}L - \overline{r}K = 0 \end{aligned} \tag{25}$$

for which the 'composite' function is

$$\phi = f(L,K) + \lambda\left(\overline{C} - \overline{w}L - \overline{r}K\right) \tag{26}$$

whose partial derivatives, with respect to *L*, *K* and λ, must equal zero for a maximum. Thus

$$\begin{aligned}
&\frac{\partial\phi}{\partial L} = \frac{\partial X}{\partial L} + \lambda\left(-\bar{w}\right) = 0 \qquad && \Rightarrow \lambda = \frac{\partial X / \partial L}{\bar{w}} \\
&\frac{\partial\phi}{\partial K} = \frac{\partial X}{\partial K} + \lambda\left(-\bar{r}\right) = 0 \qquad && \Rightarrow \lambda = \frac{\partial X / \partial K}{\bar{r}} \\
&\frac{\partial\phi}{\partial \lambda} = \bar{C} - \bar{w}L - \bar{r}K = 0
\end{aligned} \tag{27}$$

From $\frac{\partial\phi}{\partial L}$ and $\frac{\partial\phi}{\partial K}$ we get

$$\frac{\partial X / \partial L}{\bar{w}} = \lambda = \frac{\partial X / \partial K}{\bar{r}} \tag{28}$$

and rearranging

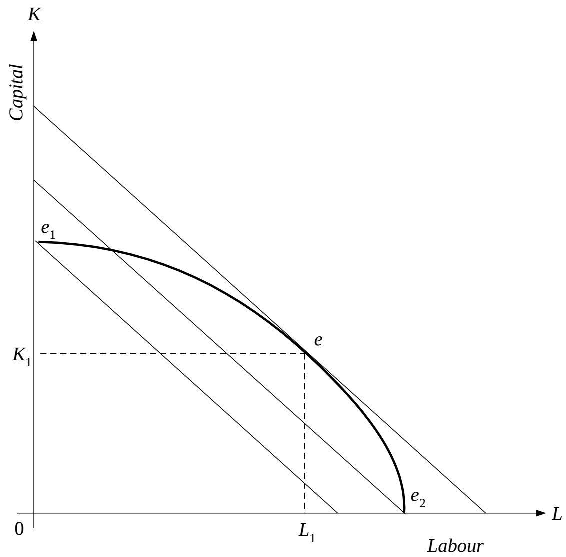
$$\frac{\partial X / \partial L}{\partial X / \partial K} = \frac{\bar{w}}{\bar{r}} = \frac{MP_L}{MP_K} \tag{29}$$

Hence, the first-order condition for a maximum is an equality of the ratios of MPs and factor prices. The second-order condition requires that the slopes of the MP curves of the factors have negative slopes. This is sufficient to ensure convex isoquants.

You should work through the case for minimising cost for a gain level of output. Be careful not to confuse these two cases.

This condition can be fulfilled however for isoquants that are concave to the origin, i.e.,

Figure 2.11 Output Maximisation & Concave Isoquants



but e is not an equilibrium since lower cost combinations e_1 and e_2 exist. This explains why there are concerns about second-order conditions. But, as noted above, the choice of functional forms in economics is almost invariably one that ensure the second-order conditions are irrelevant.

Long-Run Analysis and Returns to Scale

The distinction between long and short-run analysis is the extent to which factors can be varied. In long-run analysis all factors can be varied.

Start from the following position

$$X_0 = f(L,K) \tag{30}$$

where X_0 is a specific level of output. Now increase L and K by same constant factor k and we get

$$X^* = f(kL,kK) \tag{31}$$

If

i) $X^* = kX_0$ - constant returns to scale

ii) $X^* > kX_0$ - increasing returns to scale

iii) $X^* < kX_0$ - decreasing returns to scale.

That is simple enough but we need a bit more.

Start again from

$$X_0 = f(L,K) \tag{32}$$

and increase *L* and *K* by a constant, *k*, to give

$$X^* = f(kL,kK). \tag{33}$$

Now if *k* can be factored then we can write

$$X^* = k^{\upsilon} f(L,K) = k^{\upsilon} X_0 \tag{34}$$

where υ is <u>any</u> power. If this is the case, then the production is homogeneous, if not it is non-homogeneous.

The degree of homogeneity, υ, is a measure of returns to scale. If

i) $\upsilon = 1$ - constant returns to scale (linear homogeneous)

ii) $\upsilon > 1$ - increasing returns to scale

iii) $\upsilon < 1$ - decreasing returns to scale.

In the Cobb-Douglas case

$$X_0 = aL^{\alpha} K^{\beta} \tag{35}$$

and

$$\begin{aligned} X^* &= a(kL)^{\alpha}(kK)^{\beta} \\ &= \left(a.L^{\alpha} K^{\beta}\right) k^{(\alpha+\beta)} \\ &= k^{(\alpha+\beta)} X_0 \end{aligned} \tag{36}$$

and therefore

$$\upsilon = \alpha + \beta . \tag{37}$$

Thus

$$\alpha + \beta = 1 \Rightarrow CRTS$$

$$\alpha + \beta > 1 \Rightarrow IRTS$$

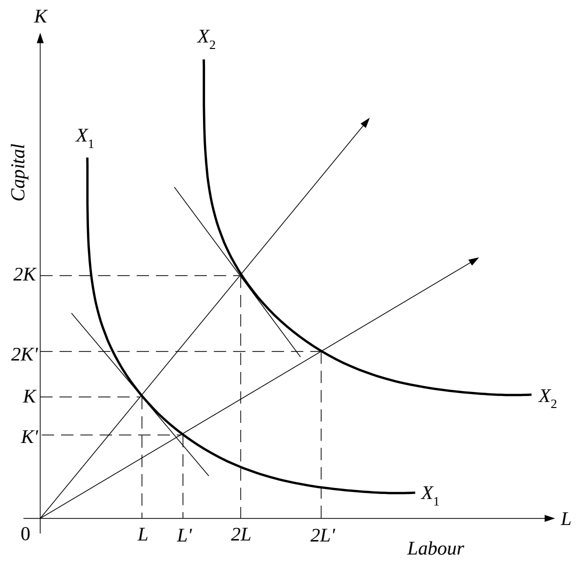
$$\alpha + \beta < 1 \Rightarrow DRTS$$

This provides us with a very simple graphical representation of returns to scale by relating the equi-proportionate change in inputs to the proportionate change in output.

i) *Constant Returns to Scale*

e.g., doubling inputs doubles output.

Figure 2.12 Constant Returns to Scale



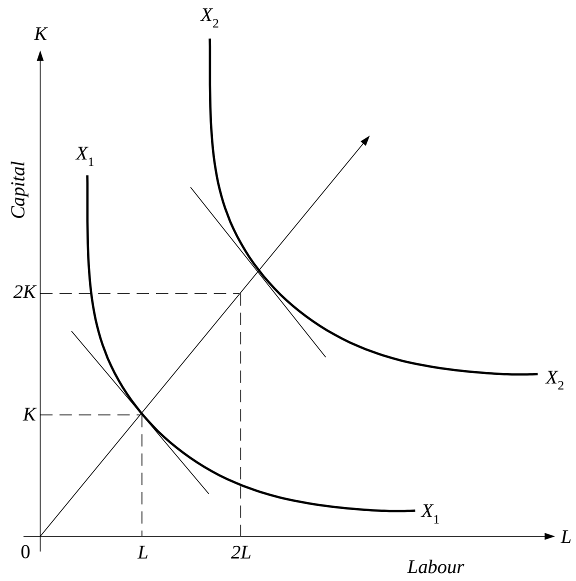
The rays are a specific type of product lines. Since all factors are variable in the long-run, they pass through the origin.

If the product lines join points on segmental isoquants where the MRTS are constant they are known as isoclines. If the isoclines are straight lines then not only is the MRTS constant but the factor (K/L) ratio is also constant, which means the production function is homogeneous. **Note:** the MRTS and factor ratio is constant for any isocline, but have different values for each isocline.

iii) *Decreasing Returns to Scale*

i.e., doubling inputs less than doubles output.

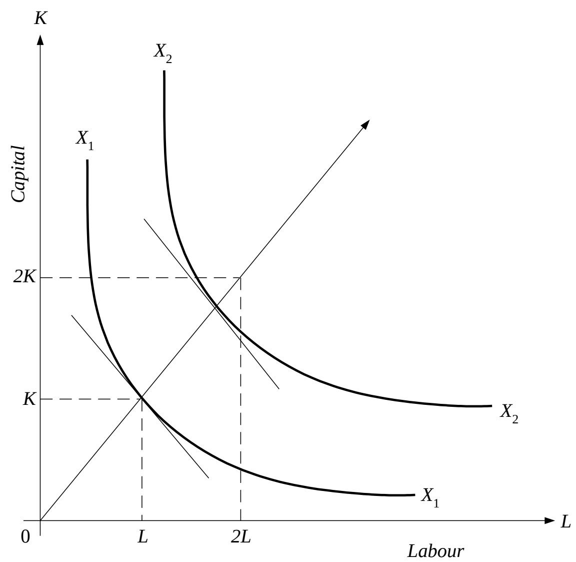
Figure 2.13 Decreasing Returns to Scale



iii) Increasing Returns to Scale

i.e., doubling inputs more than doubles output.

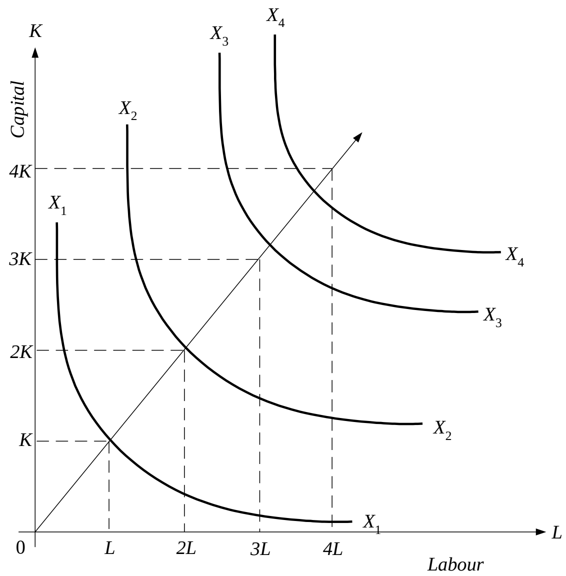
Figure 2.14 Increasing Returns to Scale



iv) Variable Returns to Scale

i.e., returns to scale not everywhere constant, decreasing or increasing.

Figure 2.15 Variable Returns to Scale



Up to X_2 RTS are constant, to X_3 we then have IRTS, and thereafter they are decreasing.

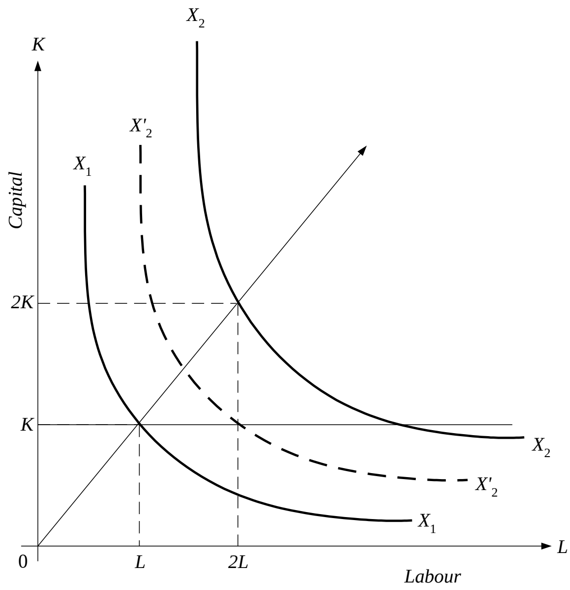
Functions with these characteristics are hard to handle and consequently economists rarely use them.

Also, non-homogeneous functions may display CRTS, DRTS or IRTS but are difficult to display. Also, the isoclines will be curves and the factor ratio will vary along each isocline. To simplify matters economists therefore prefer homogeneous functions, but it involves special and restrictive assumptions.

'Law' of Variable Proportions

The matter of returns to scale impinges upon the short-run production function since there is a relationship between isoquants and the short-run production function. To see this start with CRTS, i.e.,

Figure 2.16 Isoquants and Product Lines

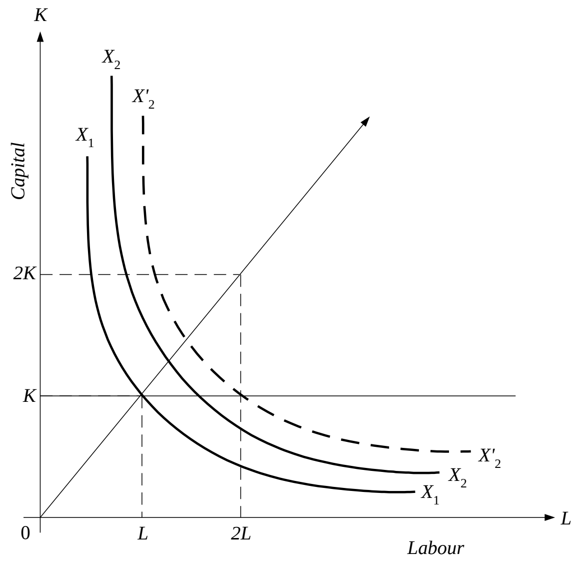


and then hold the capital input constant at *K*. Then we can draw a horizontal product line for constant, *K*, capital, and doubling labour from *L* to 2L results in a less than doubling of output. Thus, for a production function with universal CRTS, the law of diminishing returns to the variable factor holds universally. Hence, the short-run production function only exists over phase 2.

It is trivial to show that precisely the same conclusion can be derived for a production function with universal DRTS.

But, it is possible, if highly unlikely, that universal IRTS may result in the diminishing returns to the variable factor being more than offset, i.e.,

Figure 2.17 IRTS and Product Lines



To see what is happening for the ‘general’ shape of a short-run production function, you should derive an appropriate isoquant map.

Choice of Optimal Expansion Paths

The analysis of the choice of optimal input combination undertaken so far has been highly static. Conceptually, using the cost minimising case, we could argue that the firm has decided how much it wishes to produce in a period and thence maximised profits by minimising the total cost combination of inputs, i.e., chosen quantities of *L* and *K* such that

$$\text{Max} \quad \Pi = \bar{p} * \bar{X} - \bar{w}L - \bar{r}K \; . \qquad (38)$$

But in the longer term the firm would be free to vary its level of output. In which case the firm might be interested in identifying the optimal combinations of *L* and *K* for a range of output levels.

In which case, we can identify expansion paths both where all inputs are variable and where only one input is variable.

<u>Long Run Expansion Paths</u>

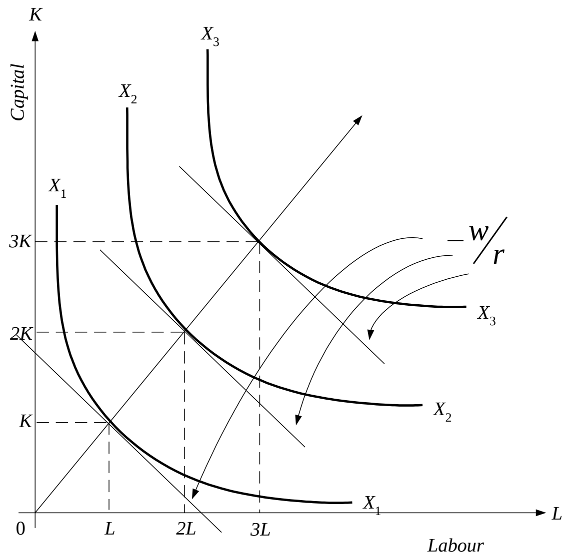
If all inputs are variable but that the output price and the input prices are fixed, the expansion path will be an isocline. This arises because along an isocline

$$MRTS_{L,K} = c = \frac{MP_L}{MP_K} \tag{39}$$

and therefore, given *w* and *r*, all points on an isocline satisfy the profit maximising criteria.

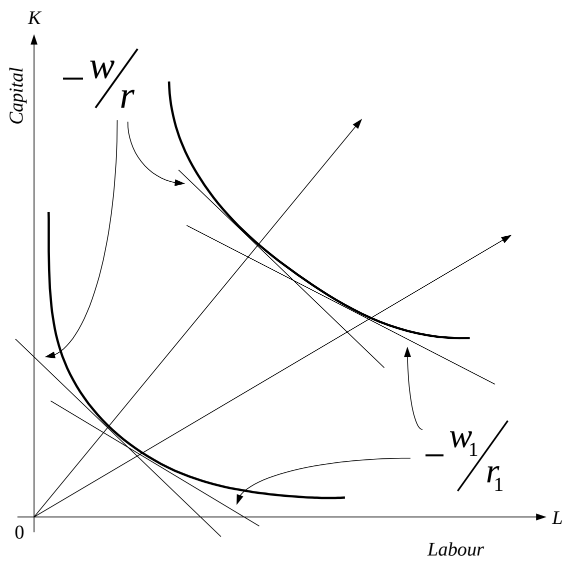
If the production function is homogeneous, then the isocline will be linear and the expansion path will be one of constant *K*/*L* ration, i.e.,

Figure 2.18 Long-run Expansion Path



and a change in relative factor prices would cause a change of expansion path, i.e.,

Figure 2.19 Long-run Expansion Path & Relative Factor Prices



But if the production function is non-homogenous the expansion path will be non-linear, and in such a case the *MRTS* is not constant and thus not associated with a constant *K*/*L* ratio.

Technology and Costs

Properties of Cobb-Douglas Production Function

$$X = a.L^{\alpha} K^{\beta} \tag{1}$$

i) *Marginal Product of Labour*

$$\begin{aligned} MP_L &= \frac{\partial X}{\partial L} = \alpha a L^{\alpha-1} K^{\beta} \\ &= \alpha \left(a L^{\alpha} K^{\beta} \right) L^{-1} \\ &= \alpha \frac{X}{L} = \alpha \left(AP_L \right) \end{aligned} \tag{2}$$

ii) $MRTS_{L,K}$

$$\begin{aligned} MRTS_{L,K} &= \frac{\partial X / \partial L}{\partial X / \partial K} \\ &= \frac{\alpha . X/L}{\beta . X/K} \\ &= \frac{\alpha}{\beta} . \frac{K}{L} \end{aligned} \tag{3}$$

iii) *Elasticity of Substitution*

$$\sigma = \frac{dK/L \big/ K/L}{dMRTS \big/ MRTS} = \frac{dK/L}{dMRTS} . \frac{MRTS}{(K/L)} \tag{4}$$

and substituting for *MRTS* gives

$$\sigma = \frac{dK/L}{d\left(\frac{\alpha}{\beta}.\left(\frac{K}{L}\right)\right)}.\frac{\left(\frac{\alpha}{\beta}.\left(\frac{K}{L}\right)\right)}{(K/L)}$$

$$= \frac{\frac{\alpha}{\beta}.d(K/L)}{d\left(\frac{\alpha}{\beta}.\left(\frac{K}{L}\right)\right).\frac{(K/L)}{(K/L)}} = \frac{\frac{\alpha}{\beta}.d\left(\frac{K}{L}\right)}{\frac{\alpha}{\beta}.d\left(\frac{K}{L}\right)} = 1 \qquad (5)$$

since α and β are constant and do not affect the derivative in the denominator. Note: other specific forms for production functions exist where $\sigma \neq 1$.

iv) Returns to Scale

Start from

$$X_0 = f(L,K) \qquad (6)$$

and increase *L* and *K* by a constant, *k*, to give

$$X^* = f(kL,kK). \qquad (7)$$

Now if *k* can be factored then we can write

$$X^* = k^\upsilon f(L,K) = k^\upsilon X_0 \qquad (8)$$

where υ is any power. In this is the case, then the production is homogeneous, if not it is non-homogeneous.

In the Cobb-Douglas case

$$X_0 = aL^\alpha K^\beta \qquad (9)$$

and

$$\begin{aligned} X^* &= a(kL)^\alpha (kK)^\beta \\ &= \left(a.L^\alpha K^\beta\right)k^{(\alpha+\beta)} \\ &= k^{(\alpha+\beta)} X_0 \end{aligned} \qquad (10)$$

and therefore

$$\upsilon = \alpha + \beta. \qquad (11)$$

Thus

i) $\alpha + \beta = 1 \Rightarrow CRTS$

ii) $\alpha+\beta>1 \Rightarrow IRTS$

iii) $\alpha+\beta<1 \Rightarrow DRTS$

Cost Minimisation in the Long-run

Formally, for a Cobb-Douglas production function

$$X = aL^{\alpha}K^{\beta} \tag{12}$$

where a and α and β are constants and (α + β) = 1, and the cost function is

$$C = wL + rK \tag{13}$$

where w and r are constant, we seek to minimize the cost of producing a given level of output.

This is simply a constrained optimisation procedure, i.e.,

$$\begin{aligned} &\text{Min} \quad C = wL + rK \\ &\text{Sto} \quad \bar{X} = aL^{\alpha}K^{\beta} \end{aligned} \tag{14a}$$

where $\bar{X}$ indicates that output is fixed.

Form the Lagrangian

$$\phi = w.L + r.K + \lambda\left(X - aL^{\alpha}K^{\beta}\right) \tag{14b}$$

and set the derivatives equal to zero

$$\begin{aligned} \frac{\partial\phi}{\partial L} &= w - \lambda\alpha aL^{\alpha-1}K^{\beta} = w - \lambda\alpha.\frac{X}{L} = 0 \\ \frac{\partial\phi}{\partial K} &= r - \lambda\beta aL^{\alpha}K^{\beta-1} = r - \lambda\beta.\frac{X}{K} = 0 \quad . \\ \frac{\partial\phi}{\partial\lambda} &= X - aL^{\alpha}K^{\beta} = 0 \end{aligned} \tag{15}$$

Solving the first two partial derivatives for λ and rearranging gives

$$\frac{w}{\alpha.\frac{X}{L}} = \lambda = \frac{r}{\beta.\frac{X}{K}} \tag{16}$$

which can be written as

$$\frac{w}{r}=\frac{\alpha.\frac{X}{L}}{\beta.\frac{X}{K}}=\frac{\alpha}{\beta}.\frac{K}{L}\,. \tag{17}$$

which is the standard first order condition where the ratio of factor prices equals the MRTS.

Starting from (17) and solving

$$K=\frac{w}{r}.\frac{\beta}{\alpha}.L \tag{18}$$

and substituting into the production constraint/function

$$X=aL^{\alpha}\left(\frac{w}{r}.\frac{\beta}{\alpha}.L\right)^{\beta} \tag{19}$$

and then solve for *L*. First rearrange the RHS

$$X=aL^{\alpha}\left(\frac{w}{r}.\frac{\beta}{\alpha}.L\right)^{\beta}=aL^{\alpha}\left(L\right)^{\beta}\left(\frac{w}{r}.\frac{\beta}{\alpha}\right)^{\beta}=aL\left(\frac{w}{r}.\frac{\beta}{\alpha}\right)^{\beta} \tag{20}$$

and then solve

$$L=\frac{1}{a\left(\frac{w}{r}.\frac{\beta}{\alpha}\right)^{\beta}}.X \tag{21}$$

which expresses the cost minimizing quantity of *L* as a function of *X*. The same can be done for *K*, i.e.,

$$K=\frac{1}{a\left(\frac{r}{w}.\frac{\alpha}{\beta}\right)^{\alpha}}.X \tag{22}$$

which expresses the cost minimizing quantity of *K* as a function of *X*.

We can now write the long run cost function as

$$\begin{aligned}C\left(X\right)&=w.L+r.K\\&=w.\frac{1}{a\left(\frac{w}{r}.\frac{\beta}{\alpha}\right)^{\beta}}+r.\frac{1}{a\left(\frac{r}{w}.\frac{\alpha}{\beta}\right)^{\alpha}}\,.\end{aligned} \tag{23}$$

If we give the production function a simple form, such as

$$X=L^{0.5}K^{05}$$

then (23) simplifies greatly, i.e.,

$$\begin{aligned} C\left(X\right) &= w.\frac{1}{\left(w.\frac{1}{r}.\frac{0.5}{0.5}\right)^{0.5}}.X + r.\frac{1}{\left(r.\frac{1}{w}.\frac{0.5}{0.5}\right)^{0.5}}.X \\ &= w.\frac{1}{w^{0.5}.r^{-0.5}}.X + r.\frac{1}{w^{-0.5}.r^{0.5}}.X \\ &= w.w^{-0.5}.r^{0.5}.X + r.w^{0.5}.r^{-0.5}.X \\ &= w^{0.5}.r^{0.5}.X + w^{0.5}.r^{0.5}.X = 2\left(w^{0.5}.r^{0.5}.X\right) \end{aligned} \quad . \tag{24}$$

Finally, it is useful to see what happens if we differentiate (24) with respect to the factor prices. These partial differentials give

$$\begin{aligned} \frac{\partial C}{\partial w} &= 2\left(0.5w^{-0.5}.r^{0.5}.X\right) = w^{-0.5}.r^{0.5}.X \\ \frac{\partial C}{\partial r} &= 2\left(0.5w^{0.5}.r^{-0.5}.X\right) = w^{0.5}.r^{-0.5}.X \end{aligned} \tag{25}$$

which, since *w*, *r* and *X* must all be positive, are both positive. Hence as factor prices increase so do costs.

Notice that these derivatives of the cost function with respect to factor prices are identical to the expressions for the cost minimizing quantities of the factors, (21) and (22), when the parameters from the specified production function are substituted. This is not a coincidence: in general the derivative of the cost function with respect to a factor price is the cost minimizing quantity of that factor.

Formal Derivation of Cost Curves from a Production Function

Formally, for a Cobb-Douglas production function

$$X = aL^{\alpha}K^{\beta} \tag{26}$$

where a and α and β are constants, and the cost function is

$$C = wL + rK \tag{27}$$

where *w* and *r* are constant, we seek to derive

$$C = f(X). \tag{28}$$

This is simply a constrained optimisation procedure, i.e.,

$$\text{Max} \quad X = aL^{\alpha}K^{\beta}$$
$$\text{Sto} \quad \overline{C} = wL + rK \tag{29}$$

where $\overline{C}$ indicates that cost is fixed.

Form the Lagrangian

$$\phi = aL^{\alpha}K^{\beta} + \lambda\left(\overline{C} - \overline{w}L - \overline{r}K\right) \tag{30}$$

and set the derivatives equal to zero

$$\begin{aligned}
\frac{\partial\phi}{\partial L} &= \frac{\partial X}{\partial L} - \lambda w = \alpha\frac{X}{L} - \lambda w = 0 \rightarrow \alpha\frac{X}{L} = \lambda w \\
\frac{\partial\phi}{\partial L} &= \frac{\partial X}{\partial K} - \lambda r = \beta\frac{X}{K} - \lambda r = 0 \rightarrow \beta\frac{X}{K} = \lambda r \\
\frac{\partial\phi}{\partial L} &= \overline{C} - wL - rK = 0
\end{aligned} \tag{31}$$

Hence

$$\lambda = \frac{\alpha.\frac{X}{L}}{w} = \frac{\beta.\frac{X}{K}}{r} \tag{32}$$

and solving

$$K = \frac{w}{r}.\frac{\beta}{\alpha}.L \tag{33}$$

Substituting for *K* in the production gives

$$\begin{aligned}
X &= aL^{\alpha}\left(\frac{w}{r}.\frac{\beta}{\alpha}.L\right)^{\beta} \\
&= a\left(\frac{w}{r}.\frac{\beta}{\alpha}\right)^{\beta} L^{(\alpha+\beta)}
\end{aligned} \tag{34}$$

and solving for *L*

$$L^{(\alpha+\beta)} = \frac{X}{a\left(\frac{w}{r}.\frac{\beta}{\alpha}\right)^{\beta}}$$

$$L = \left[\frac{X}{a\left(\frac{w}{r}.\frac{\beta}{\alpha}\right)^{\beta}}\right]^{1/(\alpha+\beta)}$$

$$L = \left(\frac{r.\alpha}{w.\beta}\right)^{\frac{\beta}{(\alpha+\beta)}}\left(\frac{X}{a}\right)^{1/(\alpha+\beta)} \qquad (35)$$

and substituting for *L* in the expression for *K* gives

$$K = \frac{w}{r}.\frac{\beta}{\alpha}\left(\left(\frac{r.\alpha}{w.\beta}\right)^{\frac{\beta}{(\alpha+\beta)}}\left(\frac{X}{b_0}\right)^{1/(\alpha+\beta)}\right)$$

$$K = \left(\frac{w.\beta}{r.\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{X}{a}\right)^{1/(\alpha+\beta)} \qquad (36)$$

Then substituting for *L* and *K* in the cost equation

$$C = w.\left[\left(\frac{r.\alpha}{w.\beta}\right)^{\frac{\beta}{(\alpha+\beta)}}\left(\frac{X}{a}\right)^{\frac{1}{(\alpha+\beta)}}\right] + r.\left[\left(\frac{w.\beta}{r.\alpha}\right)^{\frac{\alpha}{(\alpha+\beta)}}\left(\frac{X}{a}\right)^{\frac{1}{(\alpha+\beta)}}\right]$$

$$= \left(\frac{1}{a}\right)^{\frac{1}{\alpha+\beta}}\left[w.\left(\frac{r.\alpha}{w.\beta}\right)^{\frac{\beta}{(\alpha+\beta)}} + r.\left(\frac{w.\beta}{r.\alpha}\right)^{\frac{\alpha}{(\alpha+\beta)}}\right]X^{1/(\alpha+\beta)} \qquad (37)$$

Functional Forms and Cost Curves

Cost functions that produce the 'U'-shaped ATC, AVC and MC curves and an 'S'-shaped TC curves, and achieve the 'correct' intersections, are beyond the scope of this module. However, for those students interested the following functional form is among the simplest.

Let

$$C = zX \qquad (38)$$

where

$$z = AVC \qquad (39)$$

then

$$z = f\left(X\right) \tag{40}$$

Now

$$\frac{\partial C}{\partial X} = \frac{\partial\left(zX\right)}{\partial X} = z.\frac{\partial X}{\partial X} + X.\frac{\partial z}{\partial X} \tag{41}$$

by the function of a function rule. Thus

$$\frac{\partial C}{\partial X} = z + X.\frac{\partial z}{\partial X} \tag{42}$$

Then for *AVC* > 0 and *X* > 0, we get

i) if (slope of the *AVC* curve) < 0, then *MC* < *AVC*

ii) if (slope of the *AVC* curve) = 0, then *MC* = *AVC*

iii) if (slope of the *AVC* curve) > 0, then *MC* > *AVC*.

Furthermore, since the *AFC* is continuously downward sloping, the minimum point of the SATC curve is at a greater level of output than the minimum point of the SAVC curve.

The simplest total cost function that incorporates the law of variable proportion is cubic polynomial, i.e.,

$$\begin{aligned} C &= b_0 + b_1 X - b_2 X^2 + b_3 X^3 \\ TC &= TFC + TVC \end{aligned} \tag{43}$$

then

$$\begin{aligned} AVC &= \frac{TVC}{X} = b_1 - b_2 X - b_3 X^3 \\ MC &= \frac{\partial C}{\partial X} = b_1 - 2b_2 X + 3b_3 X^2 \\ ATC &= \frac{C}{X} = \frac{b_0}{X} + b_1 - b_2 X + b_3 X^3 \end{aligned} \tag{44}$$

This produces 'U'-shaped ATC, AVC and MC curves and an 'S'-shaped TC curve, and achieves the 'correct' intersections.